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Independent triangles covering given vertices of a graph

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Abstract

Let G be a simple graph of order n, k a positive integer with $n \geqslant 3k$ and X a set of any k vertices of G. We show that if the minimum degree $\delta(G) \geqslant (n+k)/2$, then G contains k independent triangles covering all vertices of X; and if the minimum degree $\delta(G) \geqslant (n+2k)/2$, then G contains k independent triangles such that each triangle contains exactly one vertex of X. The bounds on the minimum degree of G in above results are sharp. Some conjectures about independent triangles covering some given vertices are proposed.

Résumé

Soient G un graphe simple d'ordre n, k un entier positif avec $n \geqslant 3k$ et X un sous-ensemble quelconque de k sommets de G. Nous montrons que si le degré minimum $\delta(G) \geqslant (n+k)/2$, alors G contient k triangles independants couvrants tous sommets de X; et si le degré minimum $\delta(G) \geqslant (n+2k)/2$, alors G contient k triangles independants tel que tout triangle contient justement un sommet de X. Les bornes sur le degré minimum de G dans les resultats au-dessuses sont les meilleures possibles. Nous proposons également des conjectures sur les triangles independants couvrants les sommets donnés. *Mots clés*: Triangles independants; Degré minimum et graphe simple © 2001 Published by Elsevier Science B.V.

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1. Introduction and notation

In this paper, we only consider finite undirected graphs, without loops or multiple edges. We use [2] for the notation and terminology not defined here. Let G be a graph and $k \geqslant 2$ an integer. For any vertex $x \in V(G)$, we denote by $d(x)$ the number of

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neighbors of x and by $\delta(G)$ the minimum degree of G. If G has k cycles $C_1,\dots,C_k$ such that $V(C_i)\cap V(C_j)=\emptyset$ for any $i\neq j$, then we call that the cycles $C_1,\dots,C_k$ are independent, i.e., vertex-disjoint and also we call that G has k independent cycles $C_1,\dots,C_k$. K_t denotes a complete subgraph of order t in G. K_3 is also called a triangle of G. For any set S, $|S|$ denotes the number of all distinct elements in S.

Let G be a graph and let V_1 and V_2 be two subsets of vertices of G. The subgraph of G induced by the vertex set V_1 will be denoted by $G[V_1]$. For $x\in V(G)$, $N_{V_1}(x)=\{y\in V_1:\ xy\in E(G)\}$. $E(V_1,V_2)$ will denote the set of edges in G with one end-vertex in V_1 and the other in V_2, while $e(V_1,V_2)$ denotes the number of $E(V_1,V_2)$. For simplicity, let $E(x,V_2)$ ($E(V_1,x)$, respectively) stand for $E(\{x\},V_2)$ ($E(V_1,\{x\})$, respectively). Similarly, $e(x,V_1)=|E(\{x\},V_2)|$ and $e(V_1,x)=|E(V_1,\{x\})|$. Let H and K be the two subgraph of G. We use $V(H)$ and $E(H)$ to denote the vertex set of H and the edge set of H. We use $G[H]$ to denote the reduced subgraph $G[V(H)]$ by the vertex set $V(H)$. We let $E(H,K)$ and $e(H,K)$ stand for $E(V(H),V(K))$ and $e(V(H),V(K))$, respectively. Similarly, let $E(x,H)$ and $e(x,H)$ stand for $E(\{x\},V(H))$ and $e(\{x\},V(H))$, respectively. Sometimes, if T is a triangle in G and its three vertices are x, y and z, we let $xyzx$ stand for triangle T.

In 1963, Corrádi and Hajnal proved Erdös' conjecture proposed in the early 1960s which concerns independent cycles in a graph.

Theorem 1 (Corrádi and Hajnal [4]). *Let k, n be two positive integers and let G be a graph of order $n\geqslant 3k$. If the minimum degree $\delta(G)\geqslant 2k$, then G contains k independent cycles.*

In the same year (1963), Dirac obtained the following result, which was more general than Theorem 1 for the case in which k independent cycles are all triangles.

Theorem 2 (Dirac [5]). *Let k, n be two positive integers and let G be a graph of order $n\geqslant 3k$. If the minimum degree $\delta(G)\geqslant (n+k)/2$, then G contains k independent triangles.*

Brandt et al. also obtained one new result on k independent cycles which are either triangles or 4-cycles.

Theorem 3 (Brandt et al. [3]). *Let $s\leqslant k$ be two nonnegative integers and let G be a graph of order $n\geqslant 3s+4(k-s)$. If $d(x)+d(y)\geqslant n+s$ for every pair x, y of nonadjacent vertices of G, then G contains k independent cycles $C_1,\dots,C_k$ such that*

$$|V(C_i)| = 3 \quad \text{for } 1\leqslant i\leqslant s,$$
$$|V(C_j)| \leqslant 4 \quad \text{for } s< j\leqslant k.$$

Moreover, Hajnal and Szemerédi (1970) obtained the following result, which was conjectured by Erdös in the early 1960s.

Theorem 4 (Hajnal and Szemerédi [6]). *Let r, k, n be three positive integers and let G be a graph of order $n = rk$. If the minimum degree $\delta(G) \geqslant (r-1)k$, then G contains k independent K_r*'s.

For any r integers s_0, $s_1, \ldots, s_{r-1}$ and $n = s_0 + s_1 + \cdots + s_{r-1}$, we denote by $K_n(s_0, s_1, \ldots, s_{r-1})$ to be the complete r-partite graph with s_0, $s_1, \ldots, s_{r-1}$ vertices in the r various classes. In 1978, Bollobás noted that the result of Theorem 4 in the case $n = kr$ implies the general result in the case $n \geqslant kr$, i.e., he obtained the following strong result.

Theorem 5 (Bollobás [1]). *Let $K = K_n(k-1, s_1, s_2, \ldots, s_{r-1})$, where*

$$k \leqslant s_1 \leqslant \cdots \leqslant s_{r-1} \leqslant s_1 + 1, \quad n = k - 1 + \sum_{i=1}^{r-1} s_i \geqslant kr.$$

Then $K \not\supseteq k \cdot K_r$, i.e., *$K$ does not contain k independent K_r*'s, *but if G is a graph of order n and*

$$\delta(G) > \delta(K) = n - s_{r-1}$$

then $G \supseteq k \cdot K_r$, i.e., *G contains k independent K_r*'s.

Recently, many people have studied the problems of cyclability, i.e., for a given subset S of vertices, there exist a cycle or several independent cycles, covering S. We are interested in such problems, in particular the problem of covering the subset by independent triangles. We will show the followings.

Theorem 6. *Let k, n be two positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any k vertices of G. If the minimum degree $\delta(G) \geqslant (n+k)/2$, then G contains k independent triangles covering all vertices of X.*

The proof of Theorem 6 is postponed to the next section. The complete tripartite graph $K_n(k-1, (n-k)/2, (n-k+2)/2)$ has order n and $\delta(G) = (n+k)/2 - 1$. Since every triangle of this graph contains one vertex from each partite set, there is no collection of k independent triangles and, naturally, there is no collection of k independent triangles covering a set X of k vertices of G. Thus the minimum degree condition in Theorem 6 is sharp.

By using Theorem 6, we can obtain the following theorem.

Theorem 7. *Let k, n be two positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any k vertices of G. If the minimum degree $\delta(G) \geqslant (n+k)/2$, then G contains p independent triangles covering all vertices of X such that*

(a) *$p \geqslant k/2$ and*

(b) *each triangle contains at most two vertices of X.*

Theorem 6 implies directly the followings.

Corollary 8. *Let k, n be two positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any k independent vertices of G. If the minimum degree $\delta(G) \geqslant (n+k)/2$, then G contains k independent triangles such that each triangle contains exactly one vertex of X.*

We make the following conjecture to improve the above results.

Conjecture 9. *Let k, n be two positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any k vertices of G. If the minimum degree $\delta(G) \geqslant (n+k)/2$, then G contains k independent triangles covering all vertices of X such that each triangle contains at most two vertices of X.*

Under the conditions of Theorem 6, it is nature to ask if G contains k independent triangles such that each triangle contains exactly one vertex of X. It is false for general integer k if $(n+8)/5 \leqslant k \leqslant n/3$. Let us see the following examples. For $n \geqslant 12$ and $1 \leqslant t \leqslant (2n-8)/(n+4)$, put $G^* = K_{2k-2} \oplus (K_{k-1} \cup K_{n-3k+3})$, where "$\oplus$" is the join operation between two vertex-disjoint subgraphs K_{2k-2} and $K_{k-1} \cup K_{n-3k+3}$ of G and $(n+8)/(6-t) \leqslant k \leqslant n/(2+t)$. Let $X = V(K_{k-1}) \cup \{x\}$, where x is an arbitrary vertex in K_{2k-2}, then $\delta(G) = \min\{3k-4, n-k\} \geqslant (n+tk)/2$, but G^* contains no k independent triangles such that each triangle contains exactly one vertex of X.

It would be interesting to know if a graph G of order $n \geqslant 3k$ contains k independent triangles such that each triangle contains exactly one vertex of X under the conditions $1 \leqslant k < (n+8)/(6-t)$ or $n/(2+t) < k \leqslant n/3$, where $1 \leqslant t \leqslant (2n-8)/(n+4)$, and the minimum degree $\delta(G) \geqslant (n+tk)/2$.

We note that $1 \leqslant t \leqslant (2n-8)/(n+4) < 2$ in the above example graph G^* and we can obtain the following result, whose proof is postponed to the next section.

Theorem 10. *Let k, n be two positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any k vertices of G. If the minimum degree $\delta(G) \geqslant (n+2k)/2$, then G contains k independent triangles such that each triangle contains exactly one vertex of X.*

By Theorem 10, we easily obtain the following result.

Corollary 11. *Let l, k, n be three positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any l vertices of G with $l \leqslant k/2$. If the minimum degree $\delta(G) \geqslant (n+k)/2$, then G has l independent triangles such that each triangle contains exactly one vertex of X.*

When we consider the maximum number of the independent triangles covering the vertices of given vertex set in G with $\delta(G) \geqslant (n+k)/2$, we propose the following conjecture.

Conjecture 12. *Let l, k, n be three positive integers and let G be a graph of order $n \geqslant 3k$, X a set of any l vertices of G with $l \leqslant k/2$. If the minimum degree $\delta(G) \geqslant$*

$(n+k)/2$, *then* G *has* k *independent triangles* $C_1, \ldots, C_k$ *satisfying*

$$|V(C_i) \cap X| = 1 \quad \text{for any } i \in \{1, 2, \ldots, l\}.$$

Another related conjecture was proposed by Wang who proved this conjecture for the case $k = 2$ in [8].

Conjecture 13. *For each integer* $k \geqslant 2$, *there exists* $N(k)$ *such that if* G *is a graph of order* $n \geqslant N(k)$ *and the minimum degree* $\delta(G) \geqslant (n+2k)/2 - 1$, *then for any* k *independent edges* $e_1, e_2, \ldots, e_k$ *of* G, *there exist* k *vertex-disjoint cycles* $C_1, C_2, \ldots, C_k$ *in* G *such that* $e_i \in E(C_i)$ *for all* $i \in \{1, 2, \ldots, k\}$ *and* $V(C_1 \cup C_2 \cup \cdots \cup C_k) = V(G)$.

2. Proofs of Theorems 6, 7 and 10

Proof of Theorem 6. By Dirac's Theorem, G contains k independent triangles. Choose k independent triangles $T_1, T_2, \ldots, T_k$ in G such that $\bigcup_{i=1}^{k} V(T_i)$ contains as many as possible vertices of X. Suppose that there exists some $x \in X - \bigcup_{i=1}^{k} V(T_i)$.

Let $\mathscr{C} = \{T_1, T_2, \ldots, T_k\}$. For $0 \leqslant i \leqslant 3$, put $\mathscr{C}_i = \{T\colon\ T \in \mathscr{C}$ satisfying $|T \cap X| = i\}$ and $s_i = |\mathscr{C}_i|$. A triangle T in $\mathscr{C}_i$ will be called an i-triangle. Thus, we have $s_0 + s_1 + s_2 + s_3 = k$. Since $x \in X - \bigcup_{i=1}^{k} V(T_i)$, we obtain $s_1 + 2s_2 + 3s_3 \leqslant k - 1$. It follows that $n \geqslant 3k + 1$.

We distinguish the following two cases, in each of which we can obtain a contradiction.

Case 1: x has at most one neighbor in $V(G) - \bigcup_{i=1}^{k} V(T_i)$.

In this case, we first obtain the following claim.

Claim 1. (1) $e(x, T) \leqslant 2$ *for any* $T \in \mathscr{C}_i$ *with* $i \leqslant 2$;

(2) $e(x, T) \leqslant 1$ *for any* $T \in \mathscr{C}_0$.

Proof of Claim 1. If there exists some triangle $T := t_1t_2t_3t_1 \in \mathscr{C}_i$ with $i \leqslant 2$ such that $e(x, T) = 3$, let $t_2 \notin X$ and we put $T' := t_1xt_3t_1$; If there exists some triangle $T := t_1t_2t_3t_1 \in \mathscr{C}_0$ such that $e(x, T) \geqslant 2$, let $t_1, t_2 \in N(x)$ and we put $T' := t_1xt_2t_1$. Then, $\mathscr{C}' := (\mathscr{C} - \{T\}) \cup \{T'\}$ covers more vertices of X than $\mathscr{C}$, a contradiction. □

By the assumption that x has at most one neighbor in $G - \bigcup_{i=1}^{k} V(T_i)$ and by using Claim 1, we deduce

$$\begin{aligned} d(x) &\leqslant s_0 + 2s_1 + 2s_2 + 3s_3 + 1 \\ &\leqslant k + s_1 + 2s_2 + 3s_3 + 1 \\ &\leqslant k + (k-1) + 1 \\ &= \frac{k + (3k+1)}{2} - \frac{1}{2} \end{aligned}$$

$$\leqslant \frac{n+k}{2} - \frac{1}{2},$$

a contradiction.

Case 2: x has at least two neighbors in $G - \bigcup_{i=1}^{k} V(T_i)$.

In this case, let x' and x'' be two distinct neighbors of x in $G - \bigcup_{i=1}^{k} V(T_i)$. We consider two pairs of the vertices $\{x, x'\}$ and $\{x, x''\}$ respectively. Then, we can obtain the following claim.

Claim 2. (1) *If* $T \in \mathscr{C}_0$, *then* $e(\{x, x'\}, T) \leqslant 3$ *and* $e(\{x, x''\}, T) \leqslant 3$.

(2) *If* $T \in \mathscr{C}_1$, *then* $2e(x, T) + e(\{x', x''\}, T) \leqslant 8$.

(3) *If* $T \in \mathscr{C}_2$, *then* $e(\{x, x'\}, T) \leqslant 5$ *and* $e(\{x, x''\}, T) \leqslant 5$.

Proof of Claim 2. (1) If there exists some triangle $T = t_1t_2t_3t_1 \in \mathscr{C}_0$ such that $e(\{x, x'\}, T) \geqslant 4$, then there exists some $t_i \in V(T)$ satisfying $t_i \in N_T(x) \cap N_T(x')$ for $i \in \{1, 2, 3\}$. We put $T' := xx't_ix$. Then $\mathscr{C}' := (\mathscr{C} - \{T\}) \cup \{T'\}$ covers more vertices of X than $\mathscr{C}$, a contradiction. This shows that $e(\{x, x'\}, T) \leqslant 3$. Similarly $e(\{x, x''\}, T) \leqslant 3$.

(2) It is easy to obtain $e(x, T) \leqslant 2$ for any $T \in \mathscr{C}_1$ by Claim 1. Moreover, if there exists some triangle $T := t_1t_2t_3t_1 \in \mathscr{C}_1$, where $t_1 \in X$ and $t_2, t_3 \notin X$, such that $2e(x, T) + e(\{x', x''\}, T) \geqslant 9$, then $e(x, T) = 2$ and $e(\{x', x''\}, T) \geqslant 5$. So we may assume that xt_2, xt_3
$\in E(G)$ and $xt_1 \notin E(G)$, since otherwise if $xt_1 \in E(G)$ and $xt_i \in E(G)$ for some $i \in \{2, 3\}$, we put $T' := t_1xt_it_1$ and $\mathscr{C}' := (\mathscr{C} - \{T\}) \cup \{T'\}$ which covers more vertices of X than $\mathscr{C}$, a contradiction.

Since $e(\{x', x''\}, T) \geqslant 5$, without loss of generality, we may assume $e(x'', T) = 3$ and $e(x', T) \geqslant 2$, i.e., $x''t_1$, $x''t_2$, $x''t_3 \in E(G)$ and either $x't_2 \in E(G)$ or $x't_3 \in E(G)$. We put either $T' := x't_2xx'$ and $T'' := x''t_1t_3x''$ (if $x't_2 \in E(G)$) or $T' := x't_3xx'$ and $T'' := x''t_1t_2x''$ (if $x't_3 \in E(G)$). Since $s_0 + s_1 + s_2 + s_3 = k$ and $s_1 + 2s_2 + 3s_3 \leqslant k - 1$, we obtain $s_0 \neq 0$, i.e., there exists a triangle $T_0 \in \mathscr{C}_0$. It is clear to see that $(\mathscr{C} - \{T_0, T\}) \cup \{T', T''\}$ is a set of k independent triangles, which covers more vertices of X than $\mathscr{C}$, a contradiction.

(3) If there exists some triangle $T := t_1t_2t_3t_1 \in \mathscr{C}_2$, where $t_1, t_2 \in X$ and $t_3 \notin X$, such that $e(\{x, x'\}, T) = 6$, we put $T' := xt_1t_2x$. Then $(\mathscr{C} - \{T\}) \cup \{T'\}$ is a set of k independent triangles, which covers more vertices of X than $\mathcal{C}$, a contradiction. Similarly, $e(\{x, x''\}, T) \leqslant 5$. $\square$

Noting that $x'x'' \notin E(G)$, otherwise if $x'x'' \in E(G)$, we put $T' := xx'x''x$ as in the proof (2) above and we choose a triangle $T_0 \in \mathscr{C}_0$, then $(\mathscr{C} - \{T_0\}) \cup \{T'\}$ covers more vertices of X than $\mathcal{C}$, a contradiction. Similarly, x and x' (x and x'', respectively) have no common neighbor in $G - \bigcup_{i=1}^{k} V(T_i)$. Thus, by Claim 2 and the facts mentioned above (noting xx', $xx'' \in E(G)$), we can obtain that

$$\begin{aligned} d(x) + d(x') + d(x) \ + \ d(x'') &\leqslant 2 \times 3s_0 + 8s_1 + 2 \times 5s_2 + 2 \times 6s_3 \\ &\quad + 2(n - 3k - 3) + 4 \end{aligned}$$

$$\leqslant 2(s_1 + 2s_2 + 3s_3) + 6k + 2(n - 3k) - 2$$
$$\leqslant 2(k - 1) + 2n - 2.$$

This gives

$$\delta(G) \leqslant \frac{n+k}{2} - 1,$$

a contradiction.

The proof of Theorem 6 is complete. □

Let $\mathscr{C} = \{T_1, T_2, \ldots, T_s\}$ be a set of s independent triangles. If X is a given subset of vertices and $X \subseteq \bigcup_{i=1}^{s} V(T_i)$, we call that $\mathscr{C}$ is a *triangle-cover* of X. Similarly define $\mathscr{C}_i = \{T\colon\ T \in \mathscr{C}$ satisfying $|V(T) \cap X| = i\}$ and $s_i(\mathscr{C}) = |\mathscr{C}_i|$.

Proof of Theorem 7. By Theorem 6, G contains k independent triangles covering all the vertices of X, i.e., G has a triangle-cover of X. Now we choose a triangle-cover $\mathscr{C} = \{T_1, T_2, \ldots, T_s\}$ of X satisfying

(1) $s \leqslant k$ and $T_i \cap X \neq \emptyset$ for $1 \leqslant i \leqslant s$ (this implies that $s_0 = 0$);

(2) Subject to (1), $s_3(\mathscr{C})$ is as small as possible among all the triangle-covers of X and

(3) Subject to (1) and (2), $s_2(\mathscr{C})$ is as large as possible among the triangle-covers of X.

If $s_3(\mathscr{C}) = 0$, we are done. Suppose, to the contrary, that $s_3(\mathscr{C}) \geqslant 1$. It is easy to obtain $s_1 + 2s_2 + 3s_3 = k$ and $s_1 + s_2 + s_3 \leqslant k - 2$ (since $s_3(\mathscr{C}) \geqslant 1$). We distinguish the following two cases, in which we obtain a contradiction.

Case 1: For some triangle $T \in \mathscr{C}_3$, there exist two vertices of T such that these two vertices of T have one common neighbor in $G - \bigcup_{T \in \mathscr{C}} V(T)$.

In this case, put the triangle $T := xx_1x_2x \in \mathscr{C}_3$ satisfying that x_1 and x_2 have a common neighbor x_3 in $G - \bigcup_{T \in \mathscr{C}} V(T)$. Clearly $x_3 \notin X$ because $\mathscr{C}$ is a triangle-cover of X. We consider the following two possibilities.

Subcase 1.1: x has at most one neighbor in $G - (\bigcup_{T \in \mathscr{C}} V(T) \cup \{x_3\})$.

In this case, we can obtain the following claim.

Claim 3. *If $T' \in \mathscr{C}_1$, then $e(x, T') \leqslant 2$.*

Proof of Claim 3. For any triangle $T' := t_1t_2t_3t_1 \in \mathscr{C}_1$, where $t_1 \in X$ and $t_2, t_3 \notin X$, if $xt_1, xt_2 \in E(G)$, put $T^* := x_1x_2x_3x_1$ and $T^{**} := xt_1t_2x$. Then, it is easy to see that $\mathscr{C}' := (\mathscr{C} - \{T, T'\}) \cup \{T^*, T^{**}\}$ is another triangle-cover of X with $s_3(\mathscr{C}') < s_3(\mathscr{C})$, a contradiction. It follows that $e(x, T') \leqslant 2$ as required. □

By the assumption of Subcase 1.1 and Claim 3 (noting $s_3 \geqslant 1$), we obtain

$$\begin{aligned} d(x) &\leqslant 2s_1 + 3s_2 + 3(s_3 - 1) + 3 + 1 \\ &\leqslant 2(s_1 + 2s_2 + 3s_3) - 3s_3 + 1 \end{aligned}$$

$$\begin{aligned}&\leqslant 2k-2\\&=\frac{k+3k}{2}-2\\&\leqslant \frac{n+k}{2}-2,\end{aligned}$$

a contradiction.

Subcase 1.2: x has at least two neighbors in $G-(\bigcup_{T\in\mathscr{C}} V(T)\cup\{x_3\})$.

In this case, let x' and x'' be the two neighbors of x in $G-(\bigcup_{T\in\mathscr{C}} V(T)\cup\{x_3\})$.

Claim 4. *If* $T'\in\mathscr{C}_1$, *then* $2e(x,T')+e(\{x',x''\},T')\leqslant 8$.

Proof of Claim 4. By a similar argument as the proof of Claim 3, it is easy to see that for any triangle $T':=t_1t_2t_3t_1\in\mathscr{C}_1$, where $t_1\in X$ and $t_2,t_3\notin X$, $\{xt_1,xt_2\}\not\subset E(G)$ and $\{xt_1,xt_3\}\not\subset E(G)$. Then $e(x,T')\leqslant 2$ and, if $e(x,T')=2$, $xt_2,xt_3\in E(G)$.

Assume now that $2e(x,T')+e(\{x',x''\},T')\geqslant 9$, then $e(x,T')=2$ and $e(\{x',x''\},T')\geqslant 5$. Without loss of generality, we may assume $e(x'',T')=3$ and $e(x',T')\geqslant 2$. Then at least one of t_2 and t_3, say t_2, is adjacent to x'. Put $T^*:=xx't_2x$, $T^{**}:=x''t_1t_3x''$ and $T^{***}:=x_1x_2x_3x_1$ and put $\mathscr{C}':=(\mathscr{C}-\{T,T'\})\cup\{T^*,T^{**},T^{***}\}$ is another triangle-cover of X with $s_3(\mathscr{C}')<s_3(\mathscr{C})$, a contradiction. □

Claim 5. *If* $T'\in\mathscr{C}_2$ *or* $T'\in\mathscr{C}_3$, *then* $2e(x,T')+e(\{x',x''\},T')\leqslant 10$.

Proof of Claim 5. If $2e(x,T')+e(\{x',x''\},T')\geqslant 11$, then $e(x,T')=3$ and $e(\{x',x''\},T')\geqslant 5$, similar to the proof of Claim 4, we can get a similar contradiction by constructing two triangles with the vertices in $T'\cup\{x,x',x''\}$. □

Claim 6. $e(\{x,x'\},\{x_1,x_2,x_3\})+e(\{x,x''\},\{x_1,x_2,x_3\})\leqslant 10$.

Proof of Claim 6. If $e(\{x,x'\},\{x_1,x_2,x_3\})+e(\{x,x''\},\{x_1,x_2,x_3\})\geqslant 11$, without loss of generality, we assume that $e(\{x,x'\},\{x_1,x_2,x_3\})=6$ and $e(\{x,x''\},\{x_1,x_2,x_3\})\geqslant 5$, then $e(\{x\},\{x_1,x_2,x_3\})=3$, $e(\{x'\},\{x_1,x_2,x_3\})=3$ and $e(\{x''\},\{x_1,x_2,x_3\})\geqslant 2$. So at least one of the two vertices x_1 and x_2, say x_2, is adjacent to x''. Put $T^*:=x'x_1x_3x'$ and $T^{**}:=xx_2x''x$ and put $\mathscr{C}':=(\mathscr{C}-\{T\})\cup\{T^*,T^{**}\}$ is another triangle-cover of X with $s_3(\mathscr{C}')<s_3(\mathscr{C})$, a contradiction. □

It is easy to show that x and x' (x and x'', respectively) have no common neighbor in $G-(\bigcup_{T\in\mathscr{C}} V(T)\cup\{x_3\})$. By using Claims 4, 5, 6 and the facts that $s_3(\mathscr{C})\geqslant 1$ and $x'x''\notin E(G)$, we obtain

$$\begin{aligned}&d(x)+d(x')+d(x)+d(x'')\\&\quad\leqslant 8s_1+10s_2+10(s_3-1)\\&\qquad+2[n-3(s_1+s_2+s_3)-3]+(10+6)\end{aligned}$$

$$= 2(s_1 + 2s_2 + 3s_3) + 2n - 2s_3$$
$$\leqslant 2k + 2n - 2,$$

which implies

$$\delta(G) \leqslant \frac{n+k}{2} - \frac{1}{2},$$

a contradiction.

Case 2: For any $T \in \mathscr{C}_3$, any two vertices of T have no common neighbor in $G - \bigcup_{T\in\mathscr{C}} V(T)$.

Under the assumption of Case 2, we have

Claim 7. *If* $T := x_1x_2x_3x_1 \in \mathscr{C}_3$ *and* $T' := y_1y_2y_3y_1 \in \mathscr{C}_1$, *where* $x_1, x_2, x_3, y_1 \in X$ *and* $y_2, y_3 \notin X$, *then* $e(T, T') \leqslant 7$ *and if the equality holds*, $N_{T'}(x_i) = N_{T'}(x_j) = \{y_1, y_2, y_3\}$ *and* $N_{T'}(x_l) = \{y_1\}$ *for* $\{i, j, l\} = \{1, 2, 3\}$; *in particular* $e(T, y_1) = 3$.

Proof of Claim 7. For any $\{p, q\} = \{2, 3\}$ and any $\{i, j, l\} = \{1, 2, 3\}$, if y_px_i, y_px_j, y_1x_l, $y_qx_l \in E(G)$, define two triangles $T^* := y_px_ix_jy_p$ and $T^{**} := y_1x_ly_qy_1$ and put $\mathscr{C}' := (\mathscr{C} - \{T, T'\}) \cup \{T^*, T^{**}\}$, which is another triangle-cover of X with $s_3(\mathscr{C}') < s_3(\mathscr{C})$, a contradiction.

Suppose that $e(T, T') \geqslant 8$, without loss of generality, we may assume $e(x_1, T') = 3$, $e(x_2, T') = 3$ and $e(x_3, T') \geqslant 2$. One of y_2 and y_3, say y_2, is adjacent to x_3. By taking $p = 2$, $q = 3$, $i = 1$, $j = 3$ and $l = 2$ in the above argument, we have a contradiction. This shows that $e(T, T') \leqslant 7$.

Assume now that $e(T, T') = 7$ and hence, without loss of generality, that $e(x_1, T') = 3$. If $y_2 \in N(x_2) \cap N(x_3)$, by taking $p = 2$, $i = 2$, $j = 3$, $q = 3$ and $l = 1$ in the above argument, we have a contradiction. Hence, $y_2 \notin N(x_2) \cap N(x_3)$. Similarly, $y_3 \notin N(x_2) \cap N(x_3)$.

If $y_dx_g, y_fx_h \in E(G)$ with $\{d, f\} = \{g, h\} = \{2, 3\}$, we deduce that $y_1x_g, y_1x_h \in E(G)$ from $e(T, T') = 7$. By taking $p = d, i = 1, j = g, q = f, l = h$ in the above argument, we have a contradiction.

The above two facts imply that if $e(T, T') = 7$, one of x_2 and x_3, say x_2, has only y_1 as a neighbor in T'. The claim follows. □

Claim 8. *For any triangle* $T \in \mathscr{C}_3$, *there exists at most one triangle* $T' \in \mathscr{C}_1$ *such that* $e(T, T') = 7$.

Proof of Claim 8. If there exist one triangle $T := x_1x_2x_3x_1 \in \mathscr{C}_3$ and two triangles $T' := y_1y_2y_3y_1 \in \mathscr{C}_1$ and $T'' := z_1z_2z_3z_1 \in \mathscr{C}_1$, where $y_1, z_1 \in X$ and $y_2, y_3, z_2, z_3 \notin X$, such that $e(T, T') = 7$ and $e(T, T'') = 7$. By Claim 6, without loss of generality, $N_{T'}(x_1) = \{y_1\}$ and $e(x_2, T') = e(x_3, T') = 3$. Again by Claim 6, $e(T, z_1) = 3$, i.e., $N_T(z_1) = \{x_1, x_2, x_3\}$. We put $T^* := z_1x_1x_2z_1$ and $T^{**} := x_3y_1y_2x_3$. It is easy to see that $\mathscr{C}' := (\mathscr{C} - \{T, T', T''\}) \cup \{T^*, T^{**}\}$ is another triangle-cover of X with $s_3(\mathscr{C}') = s_3(\mathscr{C})$ and $s_2(\mathscr{C}') > s_2(\mathscr{C})$, contrary to the choice of $\mathscr{C}$. □

Since $s_3(\mathscr{C}) \geqslant 1$ and under the assumption of Case 2, we can choose one triangle $T := xx'x''x \in \mathscr{C}_3$ such that any two vertices of T have no common neighbor in $G - \bigcup_{T\in\mathscr{C}} V(T)$. Thus, by Claims 7 and 8, it follows that

$$\begin{aligned} & d(x) + d(x') + d(x'') \\ & \quad \leqslant [7 + 6(s_1 - 1)] + 9s_2 + 9(s_3 - 1) \\ & \qquad + [n - 3(s_1 + s_2 + s_3)] + 6 \\ & \quad = 3(s_1 + 2s_2 + 3s_3) + n - 2 - 3s_3 \\ & \quad \leqslant 3k + n - 5 \end{aligned}$$

and hence

$$\begin{aligned} 6\delta(G) & \leqslant 6k + 2n - 10 \\ & \leqslant 3k + 3n - 10. \end{aligned}$$

So we obtain

$$\delta(G) \leqslant \frac{n+k}{2} - 1,$$

a contradiction.

Thus, we can obtain the cover $\mathscr{C}$ of X such that any triangle in $\mathscr{C}$ contains at most two vertices of X, which leads $s_1 + 2s_2 = k$. Moreover, we can get $|\mathscr{C}| = s_1 + s_2 \geqslant s_1 + 2s_2/2 = k/2$.

This completes the proof of Theorem 7. □

Proof of Theorem 10. Suppose there exist a graph G of order n and a set $X \subseteq V(G)$ with $|X| = k$ such that G does not contain k independent triangles, every triangle of which contains exactly one vertex of X. We assume that G is chosen such that

$$|E(G)| \text{ is maximum.}$$

We note that $G \neq K_n$, otherwise G contains k independent triangles such that each triangle contains exactly one vertex of X. Then, there exists $uv \notin E(G)$, where $u, v \in V(G)$. By the choice of G, $G^* = G + uv$ contains k independent triangles $T_1, T_2, \ldots, T_k$ such that $|V(T_i) \cap X| = 1$ for any $i \in \{1, 2, \ldots, k\}$. By the maximality of $|E(G)|$, $uv \in E(T_i)$ for some $i \in \{1, 2, \ldots, k\}$. Without loss of generality, we may assume $i = 1$, i.e., $uv \in E(T_1)$. Set $P = T_1 - uv$, then P is a simple path in G and $V(P) \cup V(T_2) \cup \cdots \cup V(T_k) \subseteq V(G)$. Let w be the vertex of $V(T_1) - \{u, v\}$. Now we consider the following two cases, in each of which we can obtain a contradiction.

Case 1: u, $v \notin X$, i.e., $w \in X$.

Firstly, it can be easily seen that u and w (v and w, respectively) have no common neighbor in $G - \bigcup_{i=1}^{k} V(T_i)$.

Claim 9. *If* $e(w, T_i) = 3$ *for some* $i \in \{2, 3, \ldots, k\}$, *then* $e(\{u, v\}, T_i) \leqslant 4$.

Proof of Claim 9. If there exists one triangle $T_i := xyzx$ satisfying $e(w, T_i) = 3$ and $e(\{u, v\}, T_i) \geqslant 5$ for some $i \in \{2, 3, \ldots, k\}$, where $x \in X$ and $y, z \notin X$. Without loss of generality, we assume $e(u, T_i) = 3$ and $e(v, T_i) \geqslant 2$. Then one of yv and zv, say $yv \in E(G)$. We put $T_1' := uxzu$ and $T_i' := ywvy$. Thus $T_1', T_2, T_3, \ldots, T_{i-1}, T_i', T_{i+1}, T_{i+2}, \ldots, T_k$ are k independent triangles in G such that each triangle contains exactly one vertex of X, a contradiction. □

Since $uv \notin E(G)$ and by Claim 9, we obtain

$$\begin{aligned} & d(w) + d(u) + d(w) + d(v) \\ & \quad \leqslant 10(k-1) + 2(n-3k) + 4 \\ & \quad = 2n + 4k - 6 \end{aligned}$$

and hence

$$\delta(G) \leqslant \frac{n+2k}{2} - \frac{3}{2},$$

a contradiction.

Case 2: One of u and v, say u belongs to X,. (i.e., $w \notin X$).

We distinguish the following two possibilities.

Subcase 2.1: $e(\{u, v\}, T_i) = 6$ for some $i \in \{2, 3, \ldots, k\}$.

Let $T_i := xyzx$ with $x \in X$, $y, z \notin X$. Then we obtain one triangle $T_i' := vxzv$ and one path $P' := wuy$ such that $|V(T_i') \cap X| = 1$. Thus we consider path $P' = wuy$ and $k-1$ independent triangles $T_2, T_3, \ldots, T_{i-1}, T_i', T_{i+1}, T_{i+2}, \ldots, T_k$. They verify that assumption of Case 1 and thus we have a contradiction.

Subcase 2.2: $e(\{u, v\}, T_i) \leqslant 5$ for any $i \in \{2, 3, \ldots, k\}$.

If u has a neighbor $w' \in G - \bigcup_{i=2}^{k} V(T_i)$ such that $w' \neq w$, then we consider two pairs vertices $\{u, w\}$ and $\{u, w'\}$, respectively. With the similar arguments as in Case 1, i.e., using $\{u, w\}$ and $\{u, w'\}$ to stand for $\{w, u\}$ and $\{w, v\}$ respectively in Case 1, we can obtain $4\delta(G) \leqslant d(u) + d(w) + d(u) + d(w') \leqslant 2n + 4k - 6$, a contradiction. Therefore we can suppose that u has no neighbor in $G - (\bigcup_{i=2}^{k} V(T_i) \cup \{w\})$.

It follows that

$$\begin{aligned} d(u) + d(v) & \leqslant 5(k-1) + (n-3k) + 2 \\ & = n + 2k - 3, \end{aligned}$$

which gives

$$\delta(G) \leqslant \frac{n+2k}{2} - \frac{3}{2},$$

a contradiction.

This completes the proof of Theorem 10. □

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